

MATH 303 - MEASURES AND INTEGRATION
FINAL EXAM STUDY GUIDE
SOLUTIONS TO PRACTICE PROBLEMS

Problem 1.

- (a) State the monotone convergence theorem.
 (b) State the dominated convergence theorem.
 (c) Use the dominated convergence theorem to give a proof of the monotone convergence theorem.
 [Hint: If $\sup_{n \in \mathbb{N}} \int_X |f_n| d\mu < \infty$, then the set $\{x \in X : f_n(x) \neq 0 \text{ for some } n\}$ is a σ -finite set. Use this to reduce to the case that the measure space is finite.]

Solution: (a) See Theorem 3.10 in the lecture notes.

(b) See Theorem 3.28.

(c) Let $0 \leq f_1 \leq f_2 \leq \dots$ be measurable functions and $f = \lim_{n \rightarrow \infty} f_n$. By monotonicity of the integral, $\lim_{n \in \mathbb{N}} \int_X f_n d\mu$ exists and is bounded above by $\int_X f d\mu$. Hence, if $\sup_{n \in \mathbb{N}} \int_X f_n d\mu = \infty$, then we also have $\int_X f d\mu = \infty$, and there is nothing to prove.

Assume $\sup_{n \in \mathbb{N}} \int_X f_n d\mu = M < \infty$. We want to show $\int_X f d\mu = M$. Let $X_0 = \{x \in X : f_n(x) \neq 0 \text{ for some } n \in \mathbb{N}\}$. Note that

$$X_0 = \bigcup_{n, m \in \mathbb{N}} \left\{ f_n > \frac{1}{m} \right\}$$

and

$$\mu \left(\left\{ f_n > \frac{1}{m} \right\} \right) \leq m \int_X f_n d\mu \leq mM < \infty$$

by Markov's inequality. Thus, X_0 is a σ -finite set. Moreover, $f, f_n = 0$ on $X \setminus X_0$, so we may focus our attention on the set X_0 .

Write $X_0 = \bigcup_{k \in \mathbb{N}} X_k$ as an increasing union of measurable sets $X_1 \subseteq X_2 \subseteq \dots$ with $\mu(X_k) < \infty$. Let $M_k = \sup_{n \in \mathbb{N}} \int_{X_k} f_n d\mu$. We claim that $M_k = \int_{X_k} f d\mu$. By monotonicity, $M_k \leq \int_{X_k} f d\mu$, so it suffices to show $\int_{X_k} f d\mu \leq M_k$. Let $s : X_k \rightarrow [0, \infty)$ be a simple function, $0 \leq s \leq f$. Define $g_n = \min\{f_n, s\}$ so that $0 \leq g_1 \leq g_2 \leq \dots$ and $\lim_{n \rightarrow \infty} g_n = s$. The simple function s is integrable since $\mu(X_k) < \infty$, so $\int_{X_k} g_n d\mu \rightarrow \int_{X_k} s d\mu$ by the dominated convergence theorem. On the other hand, by monotonicity of the integral, $\int_{X_k} g_n d\mu \leq \int_{X_k} f_n d\mu \leq M_k$. Hence, $\int_{X_k} s d\mu \leq M_k$. Taking a supremum over simple functions $0 \leq s \leq f$, we have $\int_{X_k} f d\mu \leq M_k$ by the definition of the integral.

Now we turn to the integral over X_0 . Let $s : X \rightarrow [0, \infty)$ be a simple function, $0 \leq s \leq f$. We may write $s = \sum_{j=1}^m c_j \mathbb{1}_{E_j}$ for some sets $E_1, \dots, E_m \in \mathcal{B}$ with $E_j \subseteq X_0$ and numbers $c_1, \dots, c_m \geq 0$. Then

$$\int_{X_k} s d\mu = \sum_{j=1}^m c_j \mu(E_j \cap X_k),$$

so

$$\int_X s d\mu = \sum_{j=1}^m c_j \mu(E_j) = \lim_{k \rightarrow \infty} \int_{X_k} s d\mu$$

by continuity of μ from below. Therefore, by monotonicity of the integral,

$$\int_X s \, d\mu \leq \lim_{k \rightarrow \infty} M_k.$$

Taking a supremum over simple functions $0 \leq s \leq f$,

$$\int_X f \, d\mu \leq \lim_{k \rightarrow \infty} M_k.$$

It remains to check that $M = \lim_{k \rightarrow \infty} M_k$. Note that for each $n \in \mathbb{N}$, since $\int_X f_n \, d\mu \leq M < \infty$, the dominated convergence theorem implies

$$\int_X f_n \, d\mu = \lim_{k \rightarrow \infty} \int_{X_k} f_n \, d\mu.$$

Hence,

$$M = \sup_{n \in \mathbb{N}} \int_X f_n \, d\mu = \sup_{n \in \mathbb{N}} \sup_{k \in \mathbb{N}} \int_{X_k} f_n \, d\mu = \sup_{k \in \mathbb{N}} \sup_{n \in \mathbb{N}} \int_{X_k} f_n \, d\mu = \lim_{k \rightarrow \infty} M_k$$

by the principle of iterated suprema. Thus, $\int_X f \, d\mu \leq M$ as desired.

Problem 2. Let (X, \mathcal{B}) , (Y, \mathcal{C}) , and (Z, \mathcal{D}) be measurable spaces.

- (a) Show that $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} = \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$ and that this σ -algebra is equal to the σ -algebra on $X \times Y \times Z$ generated by the family of “measurable boxes” $\{B \times C \times D : B \in \mathcal{B}, C \in \mathcal{C}, D \in \mathcal{D}\}$.
- (b) Suppose $\mu : \mathcal{B} \rightarrow [0, \infty]$, $\nu : \mathcal{C} \rightarrow [0, \infty]$, and $\rho : \mathcal{D} \rightarrow [0, \infty]$ are σ -finite measures. Show that $(\mu \times \nu) \times \rho = \mu \times (\nu \times \rho)$ and that this measure is the unique measure on $\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}$ assigning a measure of $\mu(B)\nu(C)\rho(D)$ to each measurable box $B \times C \times D$.

Solution: (a) Let $\mathcal{F} = \{B \times C \times D : B \in \mathcal{B}, C \in \mathcal{C}, D \in \mathcal{D}\}$ be the family of measurable boxes. Writing $B \times C \times D = (B \times C) \times D = B \times (C \times D)$, we see that $\mathcal{F} \subseteq (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ and $\mathcal{F} \subseteq \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$. It therefore suffices to show $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}, \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \subseteq \sigma(\mathcal{F})$. We will show $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \subseteq \sigma(\mathcal{F})$. The other inclusion is proved in exactly the same way.

Recall that $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ is defined to be the σ -algebra generated by the family of measurable rectangles $\mathcal{R} = \{E \times D : E \in \mathcal{B} \otimes \mathcal{C}, D \in \mathcal{D}\}$, so it suffices to prove $\mathcal{R} \subseteq \sigma(\mathcal{F})$. Let $\mathcal{F}' = \{E \in \mathcal{B} \otimes \mathcal{C} : E \times D \in \sigma(\mathcal{F}) \text{ for every } D \in \mathcal{D}\}$. Then \mathcal{F}' contains all measurable rectangles $B \times C$ with $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Moreover, since $\sigma(\mathcal{F})$ is a σ -algebra, the family \mathcal{F}' is also a σ -algebra (the set operations are preserved under product with $D \in \mathcal{D}$). Thus, $\mathcal{F}' = \mathcal{B} \otimes \mathcal{C}$. This proves $\sigma(\mathcal{F}) \supseteq \mathcal{R}$ as desired.

(b) We will apply the π - λ theorem. Note that the family of measurable boxes is a π -system on $X \times Y \times Z$:

$$(B_1 \times C_1 \times D_1) \cap (B_2 \times C_2 \times D_2) = (B_1 \cap B_2) \times (C_1 \cap C_2) \times (D_1 \cap D_2).$$

Moreover, given a measurable box $B \times C \times D$, the definition of a product measure provides an equality

$$\begin{aligned} ((\mu \times \nu) \times \rho)(B \times C \times D) &= (\mu \times \nu)(B \times C)\rho(D) = \mu(B)\nu(C)\rho(D) \\ &= \mu(B)(\nu \times \rho)(C \times D) = (\mu \times (\nu \times \rho))(B \times C \times D), \end{aligned}$$

so $(\mu \times \nu) \times \rho$ and $\mu \times (\nu \times \rho)$ are both examples of measures assigning a value of $\mu(B)\nu(C)\rho(D)$ to every measurable box $B \times C \times D$.

Let $\omega_1, \omega_2 : \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} \rightarrow [0, \infty]$ be any two measures assigning a measure of $\mu(B)\nu(C)\rho(D)$ to each measurable box $B \times C \times D$. We want to show $\omega_1 = \omega_2$.

Since μ, ν , and ρ are σ -finite, we may write $X = \bigcup_{n \in \mathbb{N}} X_n$, $Y = \bigcup_{n \in \mathbb{N}} Y_n$, and $Z = \bigcup_{n \in \mathbb{N}} Z_n$ with $X_1 \subseteq X_2 \subseteq \dots$, $Y_1 \subseteq Y_2 \subseteq \dots$, $Z_1 \subseteq Z_2 \subseteq \dots$, and $\mu(X_n), \nu(Y_n), \rho(Z_n) < \infty$ for every

$n \in \mathbb{N}$. Let $A_n = X_n \times Y_n \times Z_n$ for each $n \in \mathbb{N}$. Then A_n is a measurable box, so

$$(1) \quad \omega_1(A_n) = \mu(X_n)\nu(Y_n)\rho(Z_n) = \omega_2(A_n).$$

For each $n \in \mathbb{N}$, consider the family

$$\mathcal{L}_n = \{E \in \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} : \omega_1(E \cap A_n) = \omega_2(E \cap A_n)\},$$

and let $\mathcal{L} = \bigcap_{n \in \mathbb{N}} \mathcal{L}_n$. For each $n \in \mathbb{N}$, the family \mathcal{L}_n is a λ -system, since the measures $\omega_{1,n}(E) = \omega_1(E \cap A_n)$ and $\omega_{2,n}(E) = \omega_2(E \cap A_n)$ are finite measures on $(X \times Y \times Z, \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D})$ with $\omega_{1,n}(X \times Y \times Z) = \omega_{2,n}(X \times Y \times Z)$ by (1) and $\mathcal{L}_n = \{E \in \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} : \omega_{1,n}(E) = \omega_{2,n}(E)\}$. (We showed that such families of sets are always λ -systems in the exercises; see P5 on Exercise Sheet 5.) The intersection of λ -systems is again a λ -system, so \mathcal{L} is a λ -system on $X \times Y \times Z$ containing the measurable boxes. By the π - λ theorem, it follows that $\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{L}$. Then by continuity from below of ω_1 and ω_2 , we have

$$\omega_1(E) = \lim_{n \rightarrow \infty} \omega_1(E \cap A_n) = \lim_{n \rightarrow \infty} \omega_2(E \cap A_n) = \omega_2(E)$$

for every $E \in \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}$. That is, $\omega_1 = \omega_2$.

Problem 3. Let (X, \mathcal{B}, μ) be a measure space, and let $f \in L^1(\mu)$. Show that the following are equivalent:

- (i) $f = 0$ a.e.;
- (ii) $\int_X |f| d\mu = 0$;
- (iii) $\int_E f d\mu = 0$ for every $E \in \mathcal{B}$.

Solution: See P1 on Exercise Sheet 4 on Moodle.

Problem 4. Let λ be the Lebesgue measure on \mathbb{R} . For $\varepsilon > 0$, let

$$A_\varepsilon = \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}} \text{ for infinitely many } p, q \in \mathbb{Z} \text{ with } q \geq 1 \right\}.$$

Show that $\lambda(A_\varepsilon) = 0$ for every $\varepsilon > 0$.

[Note: A theorem of Dirichlet, which can be proved using the pigeonhole principle, says that every irrational number x can be approximated by rationals in such a way that $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$ for infinitely many $p, q \in \mathbb{Z}$ with $q \geq 1$. This problem shows that the exponent 2 is best possible for almost all numbers.]

Solution: For $q \in \mathbb{N}$, let

$$A_{\varepsilon,q} = \bigcup_{p=0}^q \left(\frac{p}{q} - \frac{1}{q^{2+\varepsilon}}, \frac{p}{q} + \frac{1}{q^{2+\varepsilon}} \right).$$

Then $A_\varepsilon = \{x \in [0, 1] : x \in A_{\varepsilon,q} \text{ for infinitely many } q \in \mathbb{N}\}$. By direct computation, $\lambda(A_{\varepsilon,q}) \leq (q+1)\frac{2}{q^{2+\varepsilon}} = \frac{2}{q^{1+\varepsilon}} + \frac{1}{q^{2+\varepsilon}}$, so $\sum_{q=1}^{\infty} \lambda(A_{\varepsilon,q}) < \infty$. Hence, by the Borel–Cantelli lemma, $\lambda(A_\varepsilon) = 0$.

Problem 5. Let $a \in (0, 1)$ and $\varepsilon > 0$.

- (a) Show that there exists $M \in \mathbb{N}$ with the following property: if (X, \mathcal{B}, μ) is a probability space and $(A_n)_{n \in \mathbb{N}}$ is a sequence of measurable sets such that $\inf_{n \in \mathbb{N}} \mu(A_n) = a$, then there exist $1 \leq n < m \leq M$ such that $\mu(A_n \cap A_m) > a^2 - \varepsilon$.
- (b) Prove the following generalization for intersections of more sets. Let $k \in \mathbb{N}$. Show that there exists $M_k \in \mathbb{N}$ (depending also on a and ε) with the property: if (X, \mathcal{B}, μ) is a probability space and $(A_n)_{n \in \mathbb{N}}$ is a sequence of measurable sets such that $\inf_{n \in \mathbb{N}} \mu(A_n) = a$, then there exist

$1 \leq n_1 < n_2 < \cdots < n_k \leq M_k$ such that

$$\mu \left(\bigcap_{j=1}^k A_{n_j} \right) > a^k - \varepsilon.$$

Solution: Part (a) is a special case of (b), so we will go directly to proving (b). Fix a large value of M to be determined later. Given a probability space (X, \mathcal{B}, μ) and measurable sets $(A_n)_{n \in \mathbb{N}}$ with $\inf_{n \in \mathbb{N}} \mu(A_n) = a$, define new sets $B_n = \left\{ \sum_{m=1}^M \mathbb{1}_{A_m} = n \right\}$ for $0 \leq n \leq M$. Then for any k , we have

$$(2) \quad \sum_{m=0}^M \binom{m}{k} \mu(B_m) = \sum_{1 \leq m_1 < \cdots < m_k \leq M} \mu(A_{m_1} \cap \cdots \cap A_{m_k}).$$

Since μ is a probability space and $B_0 \sqcup B_1 \sqcup \cdots \sqcup B_M = X$, the numbers $t_m = \mu(B_m)$ are nonnegative and sum to 1. Moreover, $x \mapsto \binom{x}{k}$ is a convex function, so by Jensen's inequality,

$$\sum_{m=0}^M \binom{m}{k} \mu(B_m) \geq \left(\sum_{m=0}^M m \mu(B_m) \right) \geq \binom{Ma}{k}.$$

There are $\binom{M}{k}$ terms on the right hand side of (2), so the average size of a k -fold intersection is at least

$$\frac{\binom{Ma}{k}}{\binom{M}{k}} = \frac{Ma(Ma-1)\cdots(Ma-k+1)}{M(M-1)\cdots(M-k+1)}.$$

For fixed a and k , this ratio converges to a^k as $M \rightarrow \infty$. Thus, choosing M large enough (depending only on a , k , and ε), we can ensure

$$\frac{\binom{Ma}{k}}{\binom{M}{k}} > a^k - \varepsilon.$$

This controls the average size of k -fold intersections, so there must be at least one k -fold intersection for which the desired inequality holds.

Problem 6. Let (X, \mathcal{B}, μ) be a measure space and $f : X \rightarrow \mathbb{C}$ a measurable function. Prove Chebyshev's inequality: for every $c \in (0, \infty)$ and $p \in [1, \infty)$,

$$\mu(\{|f| > c\}) \leq \left(\frac{\|f\|_p}{c} \right)^p$$

Solution: Let $g = c\mathbb{1}_{\{|f| > c\}}$. Then $|f| \geq g$, so by monotonicity of the integral,

$$\|f\|_p^p = \int_X |f|^p d\mu \geq \int_X g^p d\mu = c^p \mu(\{|f| > c\}).$$

Dividing both sides by c^p gives Chebyshev's inequality.

Problem 7. Let $E \subseteq \mathbb{R}$ be a Lebesgue-measurable set with $\lambda(E) > 0$. Fix a finite set $F \subseteq \mathbb{R}$. Show that E contains a homothetic copy of F , i.e. a set of the form $aF + b = \{af + b : f \in F\}$ with $a \neq 0$ and $b \in \mathbb{R}$.

[Note: There is a deep theorem in additive combinatorics, known as Szemerédi's theorem, that provides a strengthening to the conclusion; namely, one can bound the scaling factor $a > \delta$ for some δ depending on the set F and the size of the set E .]

Solution: We first prove a lemma using Littlewood's first principle.

Lemma. For any $c < 1$, there exists a bounded open interval $I \subseteq \mathbb{R}$ such that $\lambda(E \cap I) > c\lambda(I)$.

Proof of Lemma. Let $c' = \frac{2c}{1+c}$ so that $\frac{c'}{2-c'} = c$. By continuity of λ from below, the set $E_N = E \cap [-N, N]$ has $\lambda(E_N) > 0$ for some $N \in \mathbb{N}$. We may therefore assume $0 < \lambda(E) < \infty$. By Littlewood's first principle, there is a finite collection of disjoint open intervals $I_i = (a_i, b_i)$, $i = 1, \dots, n$, such that

$$\lambda\left(E \Delta \bigsqcup_{i=1}^n I_i\right) < (1 - c')\lambda(E).$$

Therefore, writing $E = (E \cap \bigsqcup_{i=1}^n I_i) \sqcup (E \setminus \bigsqcup_{i=1}^n I_i)$, we have

$$(3) \quad \lambda\left(E \cap \bigsqcup_{i=1}^n I_i\right) > c'\lambda(E).$$

Also, writing $\bigsqcup_{i=1}^n I_i = (E \cap \bigsqcup_{i=1}^n I_i) \sqcup (\bigsqcup_{i=1}^n I_i \setminus E)$, we have

$$(4) \quad \lambda\left(\bigsqcup_{i=1}^n I_i\right) < (2 - c')\lambda(E).$$

In particular, combining (3) and (4),

$$\sum_{i=1}^n \lambda(E \cap I_i) = \lambda\left(E \cap \bigsqcup_{i=1}^n I_i\right) > \frac{c'}{2 - c'}\lambda\left(\bigsqcup_{i=1}^n I_i\right) = c \sum_{i=1}^n \lambda(I_i).$$

Hence, for some $i \in \{1, \dots, n\}$, we must have $\lambda(E \cap I_i) > c\lambda(I_i)$, proving the lemma. \square

Let $k = |F|$. Applying the lemma with $c = 1 - \frac{1}{2k}$, let I be an open interval such that $\lambda(E \cap I) > (1 - \frac{1}{2k})\lambda(I)$. Suppose $t_1, \dots, t_k \in (-\delta, \delta)$ with $\delta = \frac{\lambda(I)}{2k}$. Then

$$\begin{aligned} \lambda\left(\bigcap_{j=1}^k (E - t_j)\right) &\geq \lambda\left(\bigcap_{j=1}^k ((E \cap I) - t_j)\right) \\ &\geq \lambda\left(\bigcap_{j=1}^k (I - t_j)\right) - k\lambda(I \setminus E) \\ &> \lambda\left(\bigcap_{j=1}^k \underbrace{(I \cap (I - t_j))}_{\text{measure} > \lambda(I) - \delta}\right) - \frac{\lambda(I)}{2} \\ &> \lambda(I) - k\delta - \frac{\lambda(I)}{2} = 0. \end{aligned}$$

Take $a = \frac{\delta}{\max_{f \in F} |f| + 1}$ so that $|af| < \delta$ for every $f \in F$. Therefore,

$$\lambda\left(\bigcap_{f \in F} (E - af)\right) > 0.$$

In particular, $\bigcap_{f \in F} (E - af) \neq \emptyset$, so let $b \in \bigcap_{f \in F} (E - af)$. Then $af + b \in E$ for every $f \in F$, so we have found a homothetic copy $aF + b$ in E .

Problem 8. For fixed $x \in \mathbb{R}$, let $L_x = \{(x, y) : y \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be the vertical line over x . Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection onto the second coordinate $\pi(x, y) = y$. Define

$$\tau = \{G \subseteq \mathbb{R}^2 : \pi(G \cap L_x) \text{ is open for every } x \in \mathbb{R}\}.$$

- (a) Show that τ is a topology on \mathbb{R}^2 and (\mathbb{R}^2, τ) is a locally compact Hausdorff space.
 (b) Prove that $K \subseteq \mathbb{R}^2$ is compact (with respect to τ) if and only if $\pi(K \cap L_x)$ is compact for every $x \in \mathbb{R}$ and $K \cap L_x = \emptyset$ for all but finitely many x .
 (c) Define $\varphi : C_c(\mathbb{R}^2, \tau) \rightarrow \mathbb{C}$ by

$$\varphi(f) = \sum_{x \in \mathbb{R}} \int_{\mathbb{R}} f(x, y) dy,$$

where the integral with respect to y is the Riemann integral. Show that φ is a positive linear functional.

- (d) Determine the measure μ representing φ .

Solution: (a) We check that τ satisfies the axioms of a topology:

- $\emptyset \in \tau$: $\pi(\emptyset \cap L_x) = \emptyset$ for every $x \in \mathbb{R}$, and \emptyset is open in \mathbb{R} .
- $\mathbb{R}^2 \in \tau$: $\pi(\mathbb{R}^2 \cap L_x) = \mathbb{R}$ for every $x \in \mathbb{R}$ and \mathbb{R} is open in \mathbb{R} .
- τ is closed under arbitrary unions: if $(G_i)_{i \in I}$ is a family of elements of τ and $G = \bigcup_{i \in I} G_i$, then $\pi(G \cap L_x) = \bigcup_{i \in I} \pi(G_i \cap L_x)$ is open in \mathbb{R} for every $x \in \mathbb{R}$, so $G \in \tau$.
- τ is closed under finite intersections: if $G_1, G_2 \in \tau$, then $\pi(G_1 \cap G_2 \cap L_x) = \pi(G_1 \cap L_x) \cap \pi(G_2 \cap L_x)$ is open in \mathbb{R} for every $x \in \mathbb{R}$, so $G_1 \cap G_2 \in \tau$.

It remains to check that (\mathbb{R}^2, τ) is locally compact and Hausdorff. First we show that τ defines a Hausdorff topology on \mathbb{R}^2 . Let $x, y \in \mathbb{R}^2$, $x \neq y$. Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_1 \neq y_1$, then the sets $U = \{x_1\} \times \mathbb{R}$ and $V = \{y_1\} \times \mathbb{R}$ are open neighborhoods of x and y respectively, and $U \cap V = \emptyset$. If $x_2 \neq y_2$, we take $U = \mathbb{R} \times (x_2 - \delta, x_2 + \delta)$ and $V = \mathbb{R} \times (y_2 - \delta, y_2 + \delta)$ with $\delta < \frac{1}{2}|x_2 - y_2|$.

Finally, τ defines a locally compact topology: given $x = (x_1, x_2) \in \mathbb{R}^2$, the set $U = \{x_1\} \times (x_2 - 1, x_2 + 1)$ is open its closure $K = \{x_1\} \times [x_2 - 1, x_2 + 1]$ is compact. (Compactness of K follows from the argument we present below for part (b)).

(b) Suppose K is compact. We want to show $\pi(K \cap L_x)$ is compact for every $x \in \mathbb{R}$ and $K \cap L_x = \emptyset$ for all but finitely many x .

Fix $x \in \mathbb{R}$, and let $K_x = \pi(K \cap L_x)$. Let $(U_i)_{i \in I}$ be an open cover of K_x in \mathbb{R} . Then $K \subseteq \bigcup_{i \in I} (\{x\} \times U_i) \cup (\mathbb{R} \setminus \{x\}) \times \mathbb{R}$. Since K is compact, there is a finite subcover $K \subseteq \bigcup_{j=1}^n (\{x\} \times U_{i_j}) \cup (\mathbb{R} \setminus \{x\}) \times \mathbb{R}$. But then $K_x \subseteq \bigcup_{j=1}^n U_{i_j}$. Thus, we have extracted a finite subcover of K_x , so K_x is compact.

Now consider the family of sets $U_x = \{x\} \times \mathbb{R}$. Then $(U_x)_{x \in \mathbb{R}}$ is an open cover of K , so there exists a finite set $F \subseteq \mathbb{R}$ such that $K \subseteq \bigcup_{x \in F} U_x$. In other words, $\pi(K \cap L_x) = \emptyset$ for $x \notin F$.

Conversely, suppose K satisfies the two properties: $\pi(K \cap L_x)$ is compact for every $x \in \mathbb{R}$ and $K \cap L_x = \emptyset$ for all but finitely many x . Let $(G_i)_{i \in I}$ be a family of sets in τ such that $K \subseteq \bigcup_{i \in I} G_i$. By assumption, the set $F = \{x \in \mathbb{R} : \pi(K \cap L_x) \neq \emptyset\}$ is finite. Moreover, for each $x \in F$, the set $K_x = \pi(K \cap L_x)$ is compact, and we have an open cover $K_x \subseteq \bigcup_{i \in I} \pi(G_i \cap L_x)$. Hence, there is a finite subcover $K_x \subseteq \bigcup_{i \in J_x} \pi(G_i \cap L_x)$ for some finite set $J_x \subseteq I$. Letting $J = \bigcup_{x \in F} J_x$, we have that J is finite and $K \subseteq \bigcup_{j \in J} G_j$. Thus, K is a compact subset of (\mathbb{R}^2, τ) .

(c) Let $f \in C_c(\mathbb{R}^2, \tau)$, and let $K = \text{supp}(f)$. By (b), the set $F = \{x \in \mathbb{R} : \pi(K \cap L_x) \neq \emptyset\}$ is finite, and $K_x = \pi(K \cap L_x)$ is compact for each $x \in F$. We can therefore evaluate

$$\varphi(f) = \sum_{x \in F} \int_{K_x} f(x, y) dy$$

as a finite sum of Riemann integrals. Thus, the *a priori* infinite sum in the definition of φ is in fact finite, so φ is well-defined. Linearity of φ follows from linearity of addition and the Riemann integral, and positivity of φ follows from positivity of the Riemann integral. Therefore φ is a positive linear functional on $C_c(\mathbb{R}^2, \tau)$.

(d) We claim that the measure μ can be evaluated as follows: if $E \subseteq \mathbb{R}^2$ is a Borel set (with respect to τ), then

$$(5) \quad \mu(E) = \begin{cases} \sum_{x \in I} \lambda(E_x), & \text{if } I = \{x \in \mathbb{R} : E_x \neq \emptyset\} \text{ is countable,} \\ \infty, & \text{otherwise.} \end{cases}$$

Here, E_x is the cross-section $E_x = \{y \in \mathbb{R} : (x, y) \in E\} = \pi(E \cap L_x)$, and λ is the Lebesgue measure on \mathbb{R} .

Let $\nu(E)$ be the quantity defined by the right hand side of (5). By the uniqueness part of the Riesz representation theorem, it suffices to check that ν is a Radon measure on (\mathbb{R}^2, τ) and $\int_{\mathbb{R}^2} f d\nu = \varphi(f)$ for $f \in C_c(\mathbb{R}^2, \tau)$.

Claim 1: ν is a measure.

If $E = \emptyset$, then $E_x = \emptyset$ for every $x \in \mathbb{R}$, so $\nu(\emptyset) = 0$. Suppose $E_1, E_2, \dots \in \text{Borel}(\mathbb{R}^2, \tau)$ are pairwise disjoint, and let $E = \bigsqcup_{n \in \mathbb{N}} E_n$. Then $E_x = \bigsqcup_{n \in \mathbb{N}} (E_n)_x$ for every $x \in \mathbb{R}$. Let $I_n = \{x \in \mathbb{R} : (E_n)_x \neq \emptyset\}$ for each $n \in \mathbb{N}$, and note that $I = \bigcup_{n \in \mathbb{N}} I_n$. If I_n is uncountable for some $n \in \mathbb{N}$ (in which case $\nu(E_n) = \infty$), then I is uncountable, so $\nu(E) = \infty$. Therefore, $\nu(E) = \sum_{k=1}^{\infty} \nu(E_k)$. On the other hand, if I_n is countable for every $n \in \mathbb{N}$, then I is also countable, and we have

$$\nu(E) = \sum_{x \in I} \lambda(E_x) = \sum_{x \in I} \lambda\left(\bigsqcup_{n \in \mathbb{N}} (E_n)_x\right) = \sum_{x \in I} \sum_{n=1}^{\infty} \lambda((E_n)_x) = \sum_{n=1}^{\infty} \sum_{x \in I} \lambda((E_n)_x) = \sum_{n=1}^{\infty} \nu(E_n).$$

The interchange in the order of summation is justified by the fact that all of the quantities $\lambda((E_n)_x)$ are nonnegative. This proves that ν is a measure.

Claim 2: ν is locally finite.

Let $K \subseteq \mathbb{R}^2$ be compact (with respect to τ). Then by part (b), the set $I = \{x \in \mathbb{R} : K_x \neq \emptyset\}$ is finite, and K_x is compact for every $x \in I$. Hence, by local finiteness of the Lebesgue measure λ , we have that

$$\lambda(K) = \sum_{x \in I} \lambda(K_x)$$

is a finite sum of finite numbers, so $\lambda(K) < \infty$.

Claim 3: ν is outer regular.

Let $E \in \text{Borel}(\mathbb{R}^2, \tau)$ be an arbitrary Borel set. We want to show $\nu(E) = \inf_{U \supseteq E \text{ open}} \nu(U)$. Monotonicity of ν produces the inequality $\nu(E) \leq \inf_{U \supseteq E \text{ open}} \nu(U)$, so it suffices to check $\nu(E) \geq \inf_{U \supseteq E \text{ open}} \nu(U)$. If $\nu(E) = \infty$, there is nothing to check, so assume $\nu(E) < \infty$. That is, $I = \{x \in \mathbb{R} : E_x \neq \emptyset\}$ is countable and $\nu(E) = \sum_{x \in I} \lambda(E_x) < \infty$. Let $\varepsilon > 0$. For each $x \in I$, let $\varepsilon_x > 0$ such that $\sum_{x \in I} \varepsilon_x = \varepsilon$. (This is made possible by the fact that I is a

countable set.) By outer regularity of the Lebesgue measure, we may choose, for each $x \in I$, an open set $U_x \supseteq E_x$ such that $\lambda(U_x) < \lambda(E_x) + \varepsilon$. Let $U = \bigsqcup_{x \in I} (\{x\} \times U_x)$. Then $U \supseteq E$ is open in (\mathbb{R}^2, τ) and $\nu(U) = \sum_{x \in I} \lambda(U_x) < \nu(E) + \varepsilon$. Thus, $\nu(E) \geq \inf_{U \supseteq E \text{ open}} \nu(U)$ as desired.

Claim 4: ν is inner regular on open sets.

Let $U \subseteq \mathbb{R}^2$ be open (with respect to τ). We want to show $\nu(U) = \sup_{K \subseteq U \text{ compact}} \nu(K)$. By the definition of the topology τ , the set U_x is open for every $x \in \mathbb{R}$.

First suppose $I = \{x \in \mathbb{R} : U_x \neq \emptyset\}$ is countable. Then

$$\nu(U) = \sum_{x \in I} \lambda(E_x) = \sup_{F \subseteq I \text{ finite}} \sum_{x \in F} \lambda(E_x).$$

Let $F \subseteq I$ be an arbitrary finite subset. Let $\varepsilon > 0$ be arbitrary. By inner regularity of the Lebesgue measure, pick a compact set $K_x \subseteq E_x$ with $\lambda(K_x) > \lambda(E_x) - \frac{\varepsilon}{|F|}$ for each $x \in F$. Then $K = \bigsqcup_{x \in F} (\{x\} \times K_x)$ is compact in (\mathbb{R}^2, τ) by part (b), and

$$\nu(K) = \sum_{x \in F} \lambda(K_x) > \sum_{x \in F} \lambda(E_x) - \varepsilon.$$

Thus, $\sup_{K \subseteq U \text{ compact}} \nu(K) \geq \nu(U)$.

Now suppose $I = \{x \in \mathbb{R} : U_x \neq \emptyset\}$ is uncountable. In this case, we want to show $\sup_{K \subseteq U \text{ compact}} \nu(K) = \infty$. Note that $\lambda(U_x) > 0$ for every $x \in I$, since every open subset of the real line has positive Lebesgue measure. Therefore, $I_n = \{x \in \mathbb{R} : \lambda(U_x) > \frac{1}{n}\}$ is infinite (in fact, uncountable) for some $n \in \mathbb{N}$. By inner regularity of the Lebesgue measure, there exist compact sets $K_x \subseteq U_x$ for each $x \in I_n$ with the property $\lambda(K_x) > \frac{1}{2n}$. Given any finite set $F \subseteq I$, the set $K_F = \bigsqcup_{x \in F} (\{x\} \times K_x)$ is a compact subset of U , and

$$\nu(K_F) = \sum_{x \in F} \lambda(K_x) \geq \frac{1}{2n} |F|.$$

Therefore, $\sup_{K \subseteq U \text{ compact}} \nu(K) = \infty$ as desired.

Claims 1–4 together show that ν is a Radon measure. The next claim therefore completes the proof.

Claim 5: For every $f \in C_c(X)$,

$$\int_{\mathbb{R}^2} f \, d\nu = \varphi(f).$$

Decomposing f into its real and imaginary parts and then each of these into positive and negative parts, we may assume that f is a nonnegative real-valued function. Let $K = \text{supp}(f)$, and let $I = \{x \in \mathbb{R} : K_x \neq \emptyset\}$. By (b), I is a finite set and K_x is compact for every $x \in I$. Let $0 \leq s_1 \leq s_2 \leq \dots$ be a sequence of simple functions increasing to f . We may write $s_n = \sum_{j=1}^{k_n} c_{n,j} \mathbb{1}_{E_{n,j}}$ with $c_{n,j} \geq 0$ and $E_{n,j} \subseteq K$, since f vanishes outside of K and $0 \leq s_n \leq f$. Then

$$\int_{\mathbb{R}^2} s_n \, d\nu = \sum_{j=1}^{k_n} c_{n,j} \nu(E_{n,j}) = \sum_{j=1}^{k_n} c_{n,j} \sum_{x \in I} \lambda((E_{n,j})_x) = \sum_{x \in I} \sum_{j=1}^{k_n} c_{n,j} \lambda((E_{n,j})_x) = \sum_{x \in I} \int_{\mathbb{R}} (s_n)_x \, d\lambda.$$

Applying the monotone convergence theorem twice, we conclude

$$\int_{\mathbb{R}^2} f \, d\nu = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} s_n \, d\nu = \lim_{n \rightarrow \infty} \sum_{x \in I} \int_{\mathbb{R}} s_n(x, y) \, d\lambda(y) = \sum_{x \in I} \int_{K_x} f(x, y) \, dy = \varphi(f).$$

Problem 9. For $x \in [0, 1)$, consider the binary expansion $x = \sum_{j=1}^{\infty} \frac{a_j(x)}{2^j}$ with $a_j(x) \in \{0, 1\}$. Let $f(x) = \min\{j \in \mathbb{N} : a_j(x) = 1\}$.

- (a) Show that f is Borel-measurable.
- (b) Compute the integral of f with respect to the Lebesgue measure on $[0, 1)$.

[Note: This problem has a probabilistic interpretation. Sampling $x \in [0, 1)$ randomly according to the Lebesgue measure, the sequence $a_1(x), a_2(x), a_3(x), \dots$ is a sequence of independent fair coin flips (where we interpret 0 as tails and 1 as heads). With this interpretation, the value of $\int_{[0,1)} f \, d\lambda$ is the expected number of flips required until we see a result of heads.]

Solution: (a) The function f may be rewritten as

$$f = \mathbb{1}_{[1/2,1)} + 2\mathbb{1}_{[1/4,1/2)} + \dots = \sum_{n=1}^{\infty} n\mathbb{1}_{[2^{-n}, 2^{-(n-1)})},$$

which is the supremum of a sequence of measurable simple functions, hence measurable.

(b) By the monotone convergence theorem and the series expression for f ,

$$\int_0^1 f(x) \, dx = \sum_{n=0}^{\infty} n \left(2^{-(n-1)} - 2^{-n} \right) = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

To compute the value of this sum, we consider the power series $g(x) = \sum_{n=1}^{\infty} nx^n$. Note that $g(x)$ converges absolutely for $|x| < 1$, so we may differentiate term by term to obtain the expression

$$g(x) = x \sum_{n=1}^{\infty} nx^{n-1} = x \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) = x \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x}{(1-x)^2}.$$

Substituting $x = \frac{1}{2}$, we conclude

$$\int_0^1 f(x) \, dx = 2.$$

Problem 10. Let (X, \mathcal{B}, μ) be a measure space. Show that the following are equivalent:

- (i) μ is σ -finite;
- (ii) there exists a finite measure ν such that $\mu \approx \nu$;
- (iii) there exists a σ -finite measure ν such that $\mu \ll \nu$.

Solution: (i) \implies (ii). This is essentially contained in the proof of the Lebesgue decomposition theorem in the lecture notes. Write $\mu = \sum_{n=1}^{\infty} \mu_n$ with $\mu_n(X) < \infty$, and define $\nu = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n(\mu_n(X)+1)}$. Then $\nu(X) < \infty$ and $\mu \ll \nu$.

(ii) \implies (iii). Finite measures are σ -finite, so this implication is immediate.

(iii) \implies (i). Let ν be a σ -finite measure such that $\mu \ll \nu$. Then by the Radon–Nikodym theorem, there exists a Radon–Nikodym derivative $f = \frac{d\mu}{d\nu}$. Moreover, since ν is σ -finite, we may decompose $X = \bigsqcup_{n \in \mathbb{N}} X_n$ with $\nu(X_n) < \infty$. For $m \geq 0$, let

$$f_{m,n} = ((f - m)\mathbb{1}_{\{m < f \leq m+1\}} + \mathbb{1}_{\{f > m+1\}}) \mathbb{1}_{X_n}.$$

Then $0 \leq f_{m,n} \leq 1$, $\int_X f_{m,n} \, d\nu \leq \nu(X_n) < \infty$, and

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} f_{m,n} = \sum_{m=0}^{\infty} ((f - m)\mathbb{1}_{\{m < f \leq m+1\}} + \mathbb{1}_{\{f > m+1\}}) = f.$$

Therefore, if we define measures $\mu_{m,n}$ by $d\mu_{m,n} = f_{m,n} d\nu$, then $\mu_{m,n}$ is a finite positive measure for every m, n , and $\mu = \sum_{m,n} \mu_{m,n}$, so μ is σ -finite.

Problem 11. Let μ_1, μ_2 be finite positive measures on a measurable space (X, \mathcal{B}) . Characterize the pairs (μ_1, μ_2) for which $(\mu_1 - \mu_2)^+ = \mu_1$ and $(\mu_1 - \mu_2)^- = \mu_2$.

Solution: Let $\mu = \mu_1 - \mu_2$. We claim that $\mu^+ = \mu_1$ and $\mu^- = \mu_2$ if and only if $\mu_1 \perp \mu_2$.

First if $\mu_1 \perp \mu_2$, then μ_1 and μ_2 are mutually singular positive measures such that $\mu = \mu_1 - \mu_2$, so by uniqueness of the Jordan decomposition, $\mu_1 = \mu^+$ and $\mu_2 = \mu^-$.

Conversely, suppose $\mu^+ = \mu_1$ and $\mu^- = \mu_2$. By the Jordan decomposition theorem, the measures μ^+ and μ^- are mutually singular, so $\mu_1 \perp \mu_2$.

Problem 12. Let (X, \mathcal{B}, μ) be a finite measure space, and let $A, B \in \mathcal{B}$. Define $\nu(E) = \mu(E \cap A) - \mu(E \cap B)$ for $E \in \mathcal{B}$.

- Show that ν is a signed measure.
- Determine the Hahn decomposition of ν .
- Show that $\nu \ll \mu$.
- Compute the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Solution: (a) Since μ is a finite measure, the difference $\mu(E \cap A) - \mu(E \cap B)$ is well-defined. We must check $\nu(\emptyset) = 0$ and ν is countable additive. Both properties follow from the corresponding properties of the measure μ :

$$\nu(\emptyset) = \mu(\emptyset \cap A) - \mu(\emptyset \cap B) = \mu(\emptyset) - \mu(\emptyset) = 0,$$

and if $(E_n)_{n \in \mathbb{N}}$ are pairwise disjoint and $E = \bigsqcup_{n \in \mathbb{N}} E_n$, then

$$\nu(E) = \sum_{n=1}^{\infty} \mu(E_n \cap A) - \sum_{m=1}^{\infty} \mu(E_m \cap B) \stackrel{(*)}{=} \sum_{n=1}^{\infty} (\mu(E_n \cap A) - \mu(E_n \cap B)) = \sum_{n=1}^{\infty} \nu(E_n).$$

In step (*), the reordering of the sum is justified by noting that μ is a finite measure, so all of the sums involved are absolutely convergent and can be freely rearranged by the Riemann rearrangement theorem.

(b) We claim that $(A, X \setminus A)$ is a Hahn decomposition of ν . Indeed, if $E \in \mathcal{B}$ with $E \subseteq A$, then

$$\nu(E) = \mu(E \cap A) - \mu(E \cap B) = \underbrace{\mu(E \cap A) - \mu(E \cap (A \cap B))}_{\geq 0} - \underbrace{\mu(E \cap (B \setminus A))}_{\emptyset} \geq 0.$$

Moreover, if $E \in \mathcal{B}$ and $E \subseteq X \setminus A$, then

$$\nu(E) = \mu(\underbrace{E \cap A}_{\emptyset}) - \mu(E \cap B) = -\mu(E \cap B) \leq 0.$$

(c) Suppose $\mu(N) = 0$. We want to show that N is a ν -null set. Let $E \in \mathcal{B}$ with $E \subseteq N$. Then $\nu(E) = \mu(\underbrace{E \cap A}_{\subseteq N}) - \mu(\underbrace{E \cap B}_{\subseteq N}) = 0$, so N is ν -null as desired.

(d) We claim $\frac{d\nu}{d\mu} = \mathbb{1}_A - \mathbb{1}_B$ μ -a.e. To see this, let $E \in \mathcal{B}$. Then

$$\int_E (\mathbb{1}_A - \mathbb{1}_B) d\mu = \int_X (\mathbb{1}_A - \mathbb{1}_B) \mathbb{1}_E d\mu = \int_X (\mathbb{1}_{A \cap E} - \mathbb{1}_{B \cap E}) d\mu = \mu(A \cap E) - \mu(B \cap E) = \nu(E),$$

so $d\nu = (\mathbb{1}_A - \mathbb{1}_B) d\mu$ as claimed.

Problem 13. Let X be an LCH space, and let μ be a Radon measure on X . Show that there exists a closed set $C \subseteq X$ with the following two properties:

- (i) $\mu(X \setminus C) = 0$, and
- (ii) if $U \subseteq X$ is open and $U \cap C \neq \emptyset$, then $\mu(U) > 0$.

[Note: The set C is called the (topological) *support* of the measure μ .]

Solution: Let $\mathcal{U} = \{U \subseteq X \text{ open} : \mu(U) = 0\}$, and define $C = X \setminus \bigcup_{U \in \mathcal{U}} U$. Then C is a closed subset, and we will check that it has the desired properties.

(i) Let $V = X \setminus C = \bigcup_{U \in \mathcal{U}} U$. We want to show $\mu(V) = 0$. Since V is open and μ is a Radon measure, we have

$$\mu(V) = \sup_{K \subseteq V \text{ compact}} \mu(K)$$

by inner regularity of Radon measures on open sets. It therefore suffices to show: if $K \subseteq V$ is compact, then $\mu(K) = 0$. Let $K \subseteq V$ be compact. By compactness, we may find finitely many $U_1, \dots, U_n \in \mathcal{U}$ such that $K \subseteq \bigcup_{j=1}^n U_j$. Then by monotonicity and (countable) subadditivity of μ ,

$$\mu(K) \leq \sum_{j=1}^n \mu(U_j) = 0$$

since $U_j \in \mathcal{U}$ for each $j \in \{1, \dots, n\}$. This proves (i).

(ii) Suppose $U \subseteq X$ is open and $U \cap C \neq \emptyset$. Then $U \notin \mathcal{U}$, so $\mu(U) > 0$.

Problem 14. Let (X, \mathcal{B}, μ) be a σ -finite measure space. Suppose ν_1, ν_2 are positive measures on (X, \mathcal{B}, μ) with $\nu_1(X) = \nu_2(X) = 1$ and $\nu_1, \nu_2 \ll \mu$. Show

$$\sup_{E \in \mathcal{B}} (\nu_1(E) - \nu_2(E)) = \frac{1}{2} \int_X \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| d\mu.$$

Solution: Let $\nu = \nu_1 - \nu_2$. Finiteness of the measures ν_1 and ν_2 ensures that ν is a well-defined signed measure. Then the left hand side is equal to $\nu^+(E)$ and the right hand side is equal to

$$(6) \quad \frac{1}{2} \int_X \left| \frac{d\nu}{d\mu} \right| d\mu.$$

We showed in the last homework assignment that $\left| \frac{d\nu}{d\mu} \right|$ is the Radon–Nikodym derivative of $|\nu|$ with respect to μ . Hence, (6) is equal to

$$\frac{1}{2} \int_X \frac{d|\nu|}{d\mu} d\mu = \frac{1}{2} |\nu|(X).$$

Thus, our goal is to show $\nu^+(X) = \frac{1}{2} |\nu|(X)$. Since $|\nu| = \nu^+ + \nu^-$, this is equivalent to showing $\nu^+(X) = \nu^-(X)$, but this follows immediately from the assumption $\nu_1(X) = \nu_2(X) < \infty$:

$$\nu^+(X) - \nu^-(X) = (\nu^+ - \nu^-)(X) = \nu(X) = (\nu_1 - \nu_2)(X) = \nu_1(X) - \nu_2(X) = 0.$$

Problem 15. Let $A, B \subseteq [0, 1)$ be Lebesgue-measurable sets. For each $t \in [0, 1)$, let $B_t = \{b + t \pmod{1} : b \in B\}$. Show that there exists $t \in [0, 1)$ such that $\lambda(A \cap B_t) \geq \lambda(A)\lambda(B)$.

Solution: We will show that the average size of the intersection $\lambda(A \cap B_t) = \lambda(A)\lambda(B)$. Indeed, by Tonelli's theorem and invariance of Lebesgue measure under translation and reflection, we

have

$$\begin{aligned}
 \int_0^1 \lambda(A \cap B_t) dt &= \int_0^1 \int_0^1 \mathbb{1}_{A \cap B_t}(x) dx dt \\
 &= \int_0^1 \int_0^1 \mathbb{1}_A(x) \mathbb{1}_{B_t}(x) dx dt \\
 &= \int_0^1 \int_0^1 \mathbb{1}_A(x) \mathbb{1}_B(x - t \bmod 1) dx dt \\
 &= \int_0^1 \mathbb{1}_A(x) \left(\int_0^1 \mathbb{1}_B(x - t \bmod 1) dt \right) dx \\
 &= \int_0^1 \mathbb{1}_A(x) dx \int_0^1 \mathbb{1}_B(t) dt \\
 &= \lambda(A)\lambda(B).
 \end{aligned}$$

Subtracting $\lambda(A)\lambda(B)$ from both sides, we obtain

$$\int_0^1 (\lambda(A \cap B_t) - \lambda(A)\lambda(B)) dt = 0,$$

so there exists $t \in [0, 1)$ such that $\lambda(A \cap B_t) - \lambda(A)\lambda(B) \geq 0$. That is, $\lambda(A \cap B_t) \geq \lambda(A)\lambda(B)$.

Problem 16. Let X be a compact metric space, and let $T : X \rightarrow X$ be a continuous function. We say that a probability measure $\mu : \text{Borel}(X) \rightarrow [0, \infty]$ is T -invariant if $\mu(T^{-1}E) = \mu(E)$ for every $E \in \text{Borel}(X)$. Denote by $\mathcal{M}(X, T)$ the space of T -invariant Borel probability measures on X .

(a) Show that $\mathcal{M}(X, T)$ is a convex set.

Given a convex set C , a point $x \in C$ is an *extreme point* if the only solution to $x = ty + (1 - t)z$ for $t \in (0, 1)$ and $y, z \in C$ is $y = z = x$.

(b) Let $\mu \in \mathcal{M}(X, T)$. Show that the following are equivalent:

(i) μ is an extreme point of $\mathcal{M}(X, T)$.

(ii) if $E \in \mathcal{B}$ and $\mu(E \Delta T^{-1}E) = 0$, then $\mu(E) \in \{0, 1\}$.

[Hint for (ii) \implies (i): Suppose μ satisfies (ii), and write $\mu = t\nu_1 + (1 - t)\nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}(X, T)$ and $t \in (0, 1)$. Let $f = \frac{d\nu_1}{d\mu}$ and consider $E = \{f < 1\}$. Show that $\int_{E \setminus T^{-1}E} f d\mu = \int_{T^{-1}E \setminus E} f d\mu$ and deduce that $f = 1$ a.e.]

A measure satisfying (ii) is called *ergodic*. Let $\mathcal{E}(X, T)$ denote the set of ergodic T -invariant Borel probability measures on X .

(c) Suppose $\mu, \nu \in \mathcal{E}(X, T)$ and $\mu \neq \nu$. Show that $\mu \perp \nu$.

Solution: (a) Let $\mu, \nu \in \mathcal{M}(X, T)$ and $t \in [0, 1]$. Let $\rho = t\mu + (1 - t)\nu$. We want to show $\rho \in \mathcal{M}(X, T)$. Let $E \in \text{Borel}(X)$. Then

$$\rho(T^{-1}E) = t\mu(T^{-1}E) + (1 - t)\nu(T^{-1}E) = t\mu(E) + (1 - t)\nu(E) = \rho(E)$$

by T -invariance of μ and ν .

(b) Suppose μ is an extreme point of $\mathcal{M}(X, T)$. We want to show μ is an ergodic measure. Let $E \in \mathcal{B}$ such that $\mu(E \Delta T^{-1}E) = 0$ and suppose for contradiction that $t = \mu(E) \in (0, 1)$. Define probability measures $\mu_1, \mu_2 : \text{Borel}(X) \rightarrow [0, 1]$ by

$$\mu_1(A) = t^{-1}\mu(A \cap E) \quad \text{and} \quad \mu_2(A) = (1 - t)^{-1}\mu(A \setminus E).$$

Note that

$$\mu_1(T^{-1}A) = t^{-1}\mu(T^{-1}A \cap E) \stackrel{(*)}{=} t^{-1}\mu(T^{-1}A \cap T^{-1}E) \stackrel{(**)}{=} t^{-1}\mu(A \cap E) = \mu_1(A),$$

where in (*) we use $\mu(E \Delta T^{-1}E) = 0$ and in (**) we use T -invariance of μ . Similarly,

$$\mu_2(T^{-1}A) = (1-t)^{-1}\mu(T^{-1}A \setminus E) = (1-t)^{-1}\mu(T^{-1}A \setminus T^{-1}E) = (1-t)^{-1}\mu(A \setminus E) = \mu_2(A).$$

Thus, $\mu_1, \mu_2 \in \mathcal{M}(X, T)$, so $\mu = t\mu_1 + (1-t)\mu_2$ expresses μ as a nontrivial linear combination of elements of $\mathcal{M}(X, T)$. This contradicts the assumption that μ is an extreme point. Thus, $\mu(E) \in \{0, 1\}$ and μ is an ergodic measure.

Conversely, suppose μ is ergodic and write $\mu = t\nu_1 + (1-t)\nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}(X, T)$ and $t \in (0, 1)$. We want to show $\nu_1 = \nu_2 = \mu$. Note that $\nu_1, \nu_2 \ll \mu$. Let $f = \frac{d\nu_1}{d\mu}$, and consider $E = \{f < 1\}$. Then (as in the proof of the Radon–Nikodym theorem) the pair $(E, X \setminus E)$ is a Hahn decomposition of the signed measure $\mu - \nu_1$. Since μ and ν_1 are both T -invariant, the pair $(T^{-1}E, X \setminus T^{-1}E)$ is also a Hahn decomposition of $\mu - \nu_1$. Hence, $E \Delta T^{-1}E$ is a $(\mu - \nu_1)$ -null set by (the uniqueness part of) the Hahn decomposition theorem. In particular, $E \Delta T^{-1}E$ is μ -null. By the ergodicity assumption, it follows that $\mu(E) \in \{0, 1\}$. If $\mu(E) = 1$, then $f < 1$ μ -a.e., so $\nu_1(X) = \int_X f d\mu < \mu(X) = 1$, which is impossible, since ν_1 is a probability measure. Hence, $\mu(E) = 0$, so $f \geq 1$ μ -a.e. A similar argument shows $f \leq 1$ μ -a.e., so $f = 1$ μ -a.e. That is, $\nu_1 = \mu$. This proves that μ is an extreme point of $\mathcal{M}(X, T)$.

(c) Let $\mu, \nu \in \mathcal{E}(X, T)$ with $\mu \neq \nu$. Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν with respect to μ . Write $X = A \sqcup B$ with $A, B \in \mathcal{B}$ such that $\nu_a(B) = 0$ and $\nu_s(A) = 0$. Then (A, B) is a Hahn decomposition for the signed measure $\infty \cdot \mu - \nu$ and $\nu_a(E) = \nu(E \cap A)$ and $\nu_s(E) = \nu(E \cap B)$ for $E \in \mathcal{B}$. By T -invariance of μ and ν , the pair $(T^{-1}A, T^{-1}B)$ is also a Hahn decomposition of $\infty \cdot \mu - \nu$, so $A \Delta T^{-1}A = B \Delta T^{-1}B$ is a null set for $\infty \cdot \mu - \nu$. In particular, ν_a and ν_s are T -invariant measures. Suppose for contradiction that $\nu_a \neq 0$. Then $\rho = \frac{\nu_a}{\nu_a(X)} \in \mathcal{M}(X, T)$ and $\rho \ll \nu$ and $\rho \ll \mu$. By the argument in the last paragraph in the proof of part (b), it follows that $\rho = \nu$ and $\rho = \mu$. Hence, $\nu = \mu$. This is a contradiction, so we must have $\nu_a = 0$. That is, $\nu = \nu_s$ is singular to μ .